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New Parametrized Entangled State Representations and Their Applications

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We construct two new kinds of parametrized entangled states in two-mode Fock space. Using the technique of integration within an ordered product of operators, we prove that they span the complete space. Their applications to solving dynamic problems and finding new generalized squeezing are addressed.

KEY WORDS: parametrized entangled states; IWOP technique; generalized squeezing; application in dynamic problems.

In recent years the concept of quantum entanglement has received much attention from physicists. This concept, originated in EPR's (Einstein, Podolsky, and Rosen) famous paper (Einstein *et al.*, 1935), nowadays becomes more and more important, as it is one of the theoretical sources of quantum communication and quantum computation (Bennett *et al.*, 1993; Ekert *et al.*, 1996). A very concise and elegant introduction of quantum entanglement is shown in Ballentine (1998). In Fan and Klauder (1994), we have introduced the newly constructed EPR entangled state representation

$$|\zeta\rangle = \exp\left[-\frac{1}{2}|\zeta|^2 + \zeta a_1^\dagger + \zeta^* a_2^\dagger - a_2^\dagger a_1^\dagger\right]|00\rangle_{1,2}, \quad \zeta = \zeta_1 + i\zeta_2, \tag{0.1}$$

where $[a_i, a_j^\dagger] = \delta_{i,j}$, as $[a_1 + a_2^\dagger, a_1^\dagger + a_2] = 0$, $|\zeta\rangle$ is the common eigenvector of $a_1 + a_2^\dagger$ and $a_1^\dagger + a_2$,

$$(a_1 + a_2^\dagger)|\zeta\rangle = \zeta|\zeta\rangle, \quad (a_1^\dagger + a_2)|\zeta\rangle = \zeta^*|\zeta\rangle, \quad \zeta = |\zeta|e^{\imath\varphi}. \tag{0.2}$$

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The entangled states can also be used in developing Wigner distribution and squeezed states theory (Fan, 2001, 2002a,b; Fan and Fan, 1996) for two-mode correlated system. Another important application of $|\zeta\rangle$ is that it provides the diagonal representation for the Noh, Fougères, and Mandel phase operator (Fan and Sun, 2001; Noh *et al.*, 1992)

$$\sqrt{\frac{a_1 + a_2^\dagger}{a_1^\dagger + a_2}} = \int \frac{d^2\zeta}{\pi} e^{i\varphi} |\zeta\rangle\langle\zeta|, \tag{0.3}$$

thus the phase behavior of optical quantum states can be studied. This indicates that many properties regarding to quantum correlation should be studied in the entangled state representations. A question thus naturally arises: Are there any other entangled state representations in two-mode Fock space? The answer is affirmative. In this work we introduce a new parametrized entangled state

$$\begin{aligned} |\zeta\rangle_g = \exp\bigg\{ & -\frac{1}{2}|\zeta|^2 + (f\zeta + g\zeta^*)a_1^\dagger + (f\zeta^* - g\zeta)a_2^\dagger \\ & - fg(a_1^{\dagger 2} - a_2^{\dagger 2}) - (f^2 - g^2)a_1^\dagger a_2^\dagger \bigg\} |00\rangle, \end{aligned} \tag{0.4}$$

where f, g are two real parameters, not independent of each other. To qualify as a quantum mechanical representation, $|\zeta\rangle_g$ must satisfy the completeness relation

$$\int \frac{d^2\zeta}{\pi} |\zeta\rangle_{gg}\langle\zeta| = 1. \tag{0.5}$$

Using the normal ordering form of the two-mode vacuum projector

$$|00\rangle\langle 00| =: \exp[-a_1^\dagger a_1 - a_2^\dagger a_2\} : \tag{0.6}$$

and the technique of integration within an ordered product (IWOP) of operators (Fan *et al.*, 1987; Wiinsche, 1999) we perform the integration

$$\begin{aligned} & \int \frac{d^2\zeta}{\pi} |\zeta\rangle_{gg}\langle\zeta| \\ & =: \int \frac{d^2\zeta}{\pi} \exp\{-|\zeta|^2 + (f\zeta + g\zeta^*)\, a_1^\dagger + (f\zeta^* - g\zeta)\, a_2^\dagger - fg\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right) \\ & -(f^2 - g^2)\, a_1^\dagger a_2^\dagger\} + (f\zeta^* + g\zeta)\, a_1 + (f\zeta - g\zeta^*)\, a_2 - fg\left(a_1^2 - a_2^2\right) \\ & -(f^2 - g^2)a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\} : \\ & =: \exp\{(fa^\dagger + fb + ga - gb^\dagger)(fb^\dagger + ga^\dagger + fa - gb) \\ & - fg\left(a_1^{\dagger 2} - a_2^{\dagger 2}\right) - (f^2 - g^2)a_1^\dagger a_2^\dagger - fg\left(a_1^2 - a_2^2\right) \end{aligned} \tag{0.7}$$

$$-(f^2 - g^2)\,a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\}:$$
$$=: \exp\{(f^2 + g^2 - 1)\,(a_1^\dagger a_1 + a_2^\dagger a_2)\} := 1,$$

which indicates that the constraint of f and g is

$$f^2 + g^2 = 1. \tag{0.8}$$

Using (8), we see that $|\zeta\rangle_g$ obeys the eigenvector equations

$$[f(a_1 + a_2^\dagger) + g(a_1^\dagger - a_2)]|\zeta\rangle_g = \zeta|\zeta\rangle_g, \tag{0.9}$$

$$[f(a_1^\dagger + a_2) + g(a_1 - a_2^\dagger)]|\zeta\rangle_g = \zeta^*|\zeta\rangle_g. \tag{0.10}$$

It then follows

$$_g\langle\zeta'|[f(a_1^\dagger + a_2) + g(a_1 - a_2^\dagger)]|\zeta\rangle_g = \zeta^*{}_g\langle\zeta'|\zeta\rangle_g = \zeta^{*\prime}{}_g\langle\zeta'|\zeta\rangle_g,$$
$$_g\langle\zeta'|[f(a_1 + a_2^\dagger) + g(a_1^\dagger - a_2)]|\zeta\rangle_g = \zeta{}_g\langle\zeta'|\zeta\rangle_g = \zeta'{}_g\langle\zeta'|\zeta\rangle_g.$$

Therefore we conclude that $|\zeta\rangle_g$ is orthonormal,

$$_g\langle\zeta'|\zeta\rangle_g = \pi\delta(\zeta' - \zeta)\,\delta(\zeta'^* - \zeta^*), \tag{0.11}$$

and is really a quantum mechanical representation. Further, we can rewrite $|\zeta\rangle_g$ as

$$\begin{aligned}|\zeta\rangle_g &= \exp\left\{-\frac{|\zeta|^2}{2} + \zeta(fa_1^\dagger - ga_2^\dagger) + \zeta^*(fa_2^\dagger + ga_1^\dagger)\right.\\ &\qquad \left. -\ (fa_1^\dagger - ga_2^\dagger)(fa_2^\dagger + ga_1^\dagger)\right\}|00\rangle \\ &= \exp\left\{-\frac{|\zeta|^2}{2} + \zeta A_1^\dagger + \zeta^* A_2^\dagger - A_1^\dagger A_2^\dagger\right\}|00\rangle,\end{aligned} \tag{0.12}$$

where

$$\begin{pmatrix} A_1^\dagger \\ A_2^\dagger \end{pmatrix} = \begin{pmatrix} f & -g \\ g & f \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} \tag{0.13}$$

and Eqs. (9) and (10) become

$$(A_2^\dagger + A_1)|\zeta\rangle_g = \zeta|\zeta\rangle_g, \quad (A_2 + A_1^\dagger)|\zeta\rangle_g = \zeta^*|\zeta\rangle_g. \tag{0.14}$$

In the following we show that the introduction of $|\zeta\rangle_g$ is useful in solving some dynamic problems. For example, when two particles have different masses m_1 and m_2, one usually introduces the center-of-mass coordinate Q_c and the mass weighted relative momentum

$$Q_c = \mu_1 Q_1 + \mu_2 Q_2, \qquad P_r = \mu_2 P_1 - \mu_1 P_2, \tag{0.15}$$

where

$$\mu_i = \frac{m_i}{M}, \quad M = m_1 + m_2, \quad \mu_1 + \mu_2 = 1. \tag{0.16}$$

Q_i and P_i are related to Bose operators via

$$a_i = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m_i\omega_i}{\hbar}}Q_i + \frac{iP_i}{\sqrt{m_i\omega_i\hbar}}\right], \qquad i = 1, 2. \tag{0.17}$$

Because of $[Q_c, P_r] = 0$, there must exist their common eigenvector. To derive it, letting $\omega_i = \hbar = 1$, $M = m_1 + m_2 = 2$, then from (17) we have

$$Q_i = \frac{a_i + a_i^\dagger}{2\sqrt{\mu_i}}, \qquad P_i = -i\sqrt{\mu_i}(a_i - a_i^\dagger), \tag{0.18}$$

Identifying the two numbers

$$f = \sqrt{\frac{\mu_1}{2}} + \sqrt{\frac{\mu_2}{2}}, \qquad g = \sqrt{\frac{\mu_1}{2}} - \sqrt{\frac{\mu_2}{2}}, \tag{0.19}$$

which meets the condition $f^2 + g^2 = 1$, we can rewrite Q_c and P_r as

$$\begin{aligned}
2Q_c &= \sqrt{\mu_1}(a_1 + a_1^\dagger) + \sqrt{\mu_2}(a_2 + a_2^\dagger) \\
&= \frac{f+g}{\sqrt{2}}(a_1 + a_1^\dagger) + \frac{f-g}{\sqrt{2}}(a_2 + a_2^\dagger) = \frac{A_1 + A_1^\dagger}{\sqrt{2}} + \frac{A_2 + A_2^\dagger}{\sqrt{2}}, \\
2P_r &= 2\mu_1\mu_2\left[\frac{a_1 - a_1^\dagger}{\sqrt{\mu_1^i}} - \frac{a_2 - a_2^\dagger}{\sqrt{\mu_2^i}}\right] \\
&= \frac{2\sqrt{\mu_1\mu_2}}{i}\left[\frac{f-g}{\sqrt{2}}(a_1 - a_1^\dagger) - \frac{f+g}{\sqrt{2}}(a_2 - a_2^\dagger)\right] \\
&= (f^2 - g^2)\left[\frac{A_1 - A_1^\dagger}{\sqrt{2}i} - \frac{A_2 - A_2^\dagger}{\sqrt{2}i}\right].
\end{aligned} \tag{0.20}$$

From (14) and (20) we immediately get

$$\begin{aligned}
Q_c|\zeta\rangle_g &= \frac{1}{\sqrt{2}}\zeta_1|\zeta\rangle_g, \\
P_r|\zeta\rangle_g &= \frac{1}{\sqrt{2}}(f^2 - g^2)\zeta_2|\zeta\rangle_g.
\end{aligned} \tag{0.21}$$

On the other hand, we can introduce

$$\begin{aligned}
|\eta\rangle_g &= \exp\left\{-\frac{|\eta|^2}{2} + \eta A_1^\dagger - \eta^* A_2^\dagger + A_1^\dagger A_2^\dagger\right\}|00\rangle \\
&= \exp\left\{-\frac{|\eta|^2}{2} + (f\eta - g\eta^*)a_1^\dagger - (f\eta^* + g\eta)a_2^\dagger\right.
\end{aligned}$$

$$+ \left. (a_1^{\dagger 2} - a_2^{\dagger 2}) fg + a_1^\dagger a_2^\dagger (f^2 - g^2) \right\} |00\rangle, \tag{0.22}$$

which is also complete

$$\int \frac{d^2\zeta}{\pi} |\eta\rangle_{gg} \langle\eta| = 1. \tag{0.23}$$

Rewriting

$$Q_r = Q_1 - Q_2 = \frac{a_1 + a_1^\dagger}{2\sqrt{\mu_1}} - \frac{a_2 + a_2^\dagger}{2\sqrt{\mu_2}} = \frac{1}{f^2 - g^2} \left[\frac{A_1 + A_1^\dagger}{\sqrt{2}} - \frac{A_2 + A_2^\dagger}{\sqrt{2}} \right],$$

$$P_c = P_1 + P_2 = \frac{1}{\sqrt{2}i} [(A_1 - A_1^\dagger) + (A_2 - A_2^\dagger)], \tag{0.24}$$

we see that $|\eta\rangle$ satisfies the eigenvector equations

$$Q_r |\eta\rangle_g = \frac{\sqrt{2}\eta_1}{f^2 - g^2} |\eta\rangle_g, \qquad P_c |\eta\rangle_g = \sqrt{2}\eta_2 |\eta\rangle_g. \tag{0.25}$$

The inner product of ${}_g\langle\eta|$ and $|\zeta\rangle_g$ is

$${}_g\langle\eta|\zeta\rangle_g = \frac{1}{2} e^{[\eta^*\zeta - \eta\zeta^*]/2} = \frac{1}{2} e^{i[\eta_1\zeta_2 - \eta_2\zeta_1]}, \qquad \eta = \eta_1 + i\eta_2, \quad \zeta = \zeta_1 + i\zeta_2. \tag{0.26}$$

Equation (26) implies that $|\zeta\rangle_g$ and $|\eta\rangle_g$ are mutual conjugate. Now we seek what is the ${}_g\langle\zeta|$ representation of the inverse of Q_r, using (26), (27), and (24) we have

$$\begin{aligned}
{}_g\langle\zeta| \frac{1}{Q_r} &= \int \frac{d^2\eta}{\pi} \int \frac{d^2\zeta'}{\pi} {}_g\langle\zeta| \frac{1}{Q_r} |\eta\rangle_{gg} \langle\eta|\zeta'\rangle_{gg} \langle\zeta'| \\
&= \int\int\int\int \frac{d\eta_1\, d\eta_2\, d\zeta_1'\, d\zeta_2'}{\pi^2} \frac{f^2 - g^2}{4\sqrt{2}\eta_1} {}_g\langle\zeta'| e^{i[\eta_1(\zeta_2' - \zeta_2) - \eta_2(\zeta_1' - \zeta_1)]} \\
&= (f^2 - g^2) \int\int\int \frac{d\eta_1\, d\zeta_1',\, d\zeta_2'}{2\sqrt{2\pi}\,\eta_1} {}_g\langle\zeta'| \delta(\zeta_1' - \zeta_1) \exp\{i\eta_1(\zeta_2' - \zeta_2)\} \\
&= (f^2 - g^2)\langle 00| \int\int \frac{d\eta_1\, d\zeta_2'}{2\sqrt{2}\pi\,\eta_1} \exp\left\{ -\frac{\zeta_1^2 + \zeta_2'^2}{2} + \zeta_1(A + B) \right. \\
&\quad \left. + i\zeta_2'(B - A) + i\eta_1(\zeta_2' - \zeta_2) - AB \right\}.
\end{aligned} \tag{0.27}$$

Using the mathematical formula

$$\int_{-\infty}^{\infty} \frac{d\eta_1}{\eta_1} e^{i\eta_1(\zeta_2' - \zeta_2)} = 2\pi i \times \begin{matrix} 1, & \zeta_2' > \zeta_2 \\ 0, & \zeta_2 > \zeta_2' \end{matrix} \tag{0.28}$$

we have

$$ {}_g\langle\zeta|\frac{1}{Q_r} = \frac{i(f^2-g^2)}{\sqrt{2}}\int_{\zeta_2}^{\infty} d\zeta_2'\,{}_g\langle\zeta_1,\zeta_2'|. \tag{0.29} $$

Now we consider the 1-dimension hydrogen atom

$$ H = \frac{P_1^2}{2m_1}+\frac{P_2^2}{2m_2}+\frac{k}{Q_1-Q_2}. \tag{0.30} $$

In terms of P_c, Q_r, and P_r we recast H as

$$ H = \frac{P_c^2}{2M}+\frac{P_r^2}{2\mu}+\frac{g}{Q_r}, \tag{0.31} $$

where $\mu = \frac{m_1 m_2}{M}$. The term $P_c^2/2M$ denotes a free particle motion and need not be paid attention. In the representation of ${}_g\langle\zeta| \equiv_g \langle\zeta_1,\zeta_2|$, the energy eigenvector equation $(\frac{P_r^2}{2\mu}+\frac{k}{Q_r})|\psi\rangle = E|\psi\rangle$ takes the form

$$ \begin{aligned} &{}_g\langle\zeta_1=0,\zeta_2|\left(\frac{P_r^2}{2\mu}+\frac{k}{Q_r}\right)|\psi\rangle \\ &\quad= \frac{(f^2-g^2)^2}{4\mu}\zeta_2^2\psi(0,\zeta_2)+\frac{ik(f^2-g^2)}{\sqrt{2}}\int_{\zeta_2}^{\infty} d\zeta_2'\psi(0,\zeta_2') \\ &\quad= E\psi(0,\zeta_2). \end{aligned} \tag{0.32} $$

Consider only the bounded states for $E<0$ case, we can set $s^2 = -4\pi E/(f^2-g^2)^2$, so Eq. (33) becomes

$$ \frac{\psi(0,\zeta_2)}{\int_{\zeta_2}^{\infty} d\zeta_2'\psi(0,\zeta_2')} = -\frac{\sqrt{8}i\mu k}{(f^2-g^2)\left(\zeta_2^2+s^2\right)}. \tag{0.33} $$

Its solution is

$$ \int_{\zeta_2}^{\infty} d\zeta_2'\psi(0,\zeta_2') = \exp\left\{\frac{-\sqrt{8i\mu k}}{s(f^2-g^2)}\arctan\frac{\zeta_2}{s}\right\}+C, \tag{0.34} $$

where C is an integral constant. Supposing $\arctan\frac{\zeta_2}{s} = \theta\pm\pi$, demanded by the single-value condition of the wave function, we must set

$$ \frac{\sqrt{8}\mu k}{s(f^2-g^2)} = 2n, \qquad n: \text{ integer} \tag{0.35} $$

so that $\exp\{\frac{-\sqrt{8}i\mu k}{s(f^2-g^2)}\arctan\frac{\zeta_2}{s}\} = \exp\{-i2n(\theta\pm\pi)\} = \exp\{-2n\theta\}$. Thus the energy quantization is

$$ E = -\frac{(f^2-g^2)^2}{4\mu}s^2 = -\frac{(f^2-g^2)^2}{4\mu}\left(\frac{\sqrt{2}\mu k}{n(f^2-g^2)}\right)^2 = -\frac{\mu k^2}{2n^2}. \tag{0.36} $$

Finally, using the IWOP technique, we perform the following integration:

$$S \equiv \int \frac{d^2\zeta}{\tau\pi} |\zeta/\tau\rangle_{gg}\langle\zeta| = \sec h\lambda \; \exp[-\tanh\lambda A_1^\dagger A_2^\dagger] : \exp\{(\sec h\lambda - 1) \times (a_1^\dagger a_1 + a_2^\dagger a_2)\} : \exp[\tanh\lambda A_1 A_2] \tag{0.37}$$

where $\tau = \exp[\lambda]$. Under S transformation

$$S|\zeta\rangle_g = \frac{1}{\tau}|\zeta/\tau\rangle_g,$$

which is a generalized squeezing. One can also study the squeezing property of the new state $S|00\rangle$.

In summary, we have introduced two parametrized entangled states $|\eta\rangle_g$ and $|\zeta\rangle_g$, and applied them to deriving the energy quantization for some dynamic systems. The new parametrized entangled states in this work together with Hong-Yi and Gui-Chuan (2002) and Hong-Yi and Yue (1996) enrich quantum mechanical representation theory. As one can see, it is the IWOP technique that helps us to find these new representations.

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